MATHEMATICS OF OPERATIONS RESEARCH
Vol. 2, No. 3, August 1977
Printed in U.S.A.

GRAPHS AND COOPERATION IN GAMES*

ROGER B. MYERSON

Northwestern University

Graph-theoretic ideas are used to analyze cooperation structures in games. Allocation rules, selecting a payoff for every possible cooperation structure, are studied for games in characteristic function form. Fair allocation rules are defined, and these are proven to be unique, closely related to the Shapley value, and stable for a wide class of games.

1. Introduction. In the study of games, one often assumes either that all players will cooperate with each other, or else that the game will be played noncooperatively. However, there are many intermediate possibilities between universal cooperation and no cooperation. (See Aumann and Drèze [1974] for one systematic study of the implications of partial cooperation.) In this paper we use ideas from graph theory to provide a framework within which we can discuss a broad class of partial cooperation structures and study the question of how the outcome of a game should depend on which players cooperate with each other.

Let N be a nonempty finite set, to be interpreted as the set of players. A *graph* on N is a set of unordered pairs of distinct members of N. We will refer to these unordered pairs as *links*, and we will denote the link between n and m by $n : m$. (So $n : m = m : n$, since the link is an unordered pair.) Let g^N be the complete graph of all links:

$$g^N = \{n : m \mid n \in N, m \in N, n \neq m\}. \tag{1}$$

Then let GR be the set of all graphs on N, so that

$$GR = \{g \mid g \subseteq g^N\}. \tag{2}$$

Our basic idea is that players may cooperate in a game by forming a series of bilateral agreements among themselves. These bilateral cooperative agreements can be represented by links between the agreeing players, so any cooperation structure can be represented by a set of agreement links. In this way, we can identify the set of all possible *cooperation structures* with GR, the set of all *graphs* on the set of players.

2. Coalitions and connectedness. A *coalition* is a nonempty subset of N. We will need a few basic concepts of connectedness, to relate coalitions and cooperation graphs.

Suppose $S \subseteq N$, $g \in GR$, $n \in S$, and $m \in S$ are given. Then we say that n and m are *connected in S by g* iff there is a path in g which goes from n to m and stays within S. That is, n and m are connected in S by g if $n = m$ or if there is some $k \geqslant 1$ and a sequence $(n^0, n^1, \ldots, n^k)$ such that $n^0 = n$, $n^k = m$, and $n^{i-1} : n^i \in g$ and $n^i \in S$ for all i from 1 to k.

Given $g \in GR$ and $S \subseteq N$, there is a unique partition of S which groups players together iff they are connected in S by g. We will denote this partition by S/g (read "S divided by g"), so:

$$S/g = \{\{i \mid i \text{ and } j \text{ are connected in } S \text{ by } g\} \mid j \in S\}. \tag{3}$$

*Received September 2, 1976; revised June 2, 1977.
AMS 1970 subject classification. Primary 90D12. Secondary 05C99.
IAOR 1973 subject classification. Main: Games. Cross Reference: Graphs.
Key words. Game theory, graphs, fair allocation rule, Shapley value.

We can interpret S/g as the collection of smaller coalitions into which S would break up, if players could only coordinate along the links in g.

For example, if $N = \{1, 2, 3, 4, 5\}$ and $g = \{1:2, 1:4, 2:4, 3:4\}$ then $\{1, 2, 3\}/g = \{\{1, 2\}, \{3\}\}$ and $N/g = \{\{1, 2, 3, 4\}, \{5\}\}$.

When we speak of *connectedness* without reference to any specific coalition, we will always mean *connectedness in* N. Given a cooperation graph g, the connectedness-partition N/g is the natural coalition structure to associate with graph g. The idea is that, even if two players do not have a direct agreement link between themselves, they may still effectively cooperate if they both have an agreement with the same mutual friend or if they are otherwise connected by the cooperation graph.

3. Allocation rules. We can now turn to the question posed in the first paragraph: how will the outcome of a given game depend on the cooperation structure?

Let v be a game in characteristic function form. That is, v is a function which maps each coalition S to a real number $v(S)$. Each number $v(S)$ is interpreted as the wealth of transferrable utility which the members of S would have to divide among themselves if they were to cooperate together and with no one outside S. We can let GR be the set of all possible cooperation structures for the game v, and the outcomes of v can be represented by payoff allocation vectors in $\mathbf{R}^N$. So we can describe how the outcome of v might depend on the cooperation structure by a function $Y: GR \to \mathbf{R}^N$, mapping cooperation graphs to allocation vectors. The idea is that $Y_n(g)$ (the n-component of $Y(g)$) should be the utility payoff which player n would expect in game v if g represented pattern of cooperative agreements between the players.

Formally, we define an *allocation rule* for v to be any function $Y: GR \to \mathbf{R}^N$ such that

$$\forall g \in GR, \forall S \in N/g, \qquad \sum_{n \in S} Y_n(g) = v(S). \tag{4}$$

This condition (4) asserts that, if S is a connected component of g, then the members of S ought to allocate to themselves the total wealth $v(S)$ available to them. This expresses our idea that N/g is the natural coalition structure to associate with a cooperation graph g. Notice that the allocation within a connected coalition S still depends on the actual graph g. For example, an allocation rule might give higher payoff to player 1 in $g_1 = \{1:2, 1:3, 1:4\}$ than in $g_2 = \{1:2, 2:3, 3:4\}$, because 1's position is more essential to coordinating the others in g_1. In each case, however, condition (4) requires that $\sum_{n=1}^{4} Y_n(g_1) = v(\{1, 2, 3, 4\}) = \sum_{n=1}^{4} Y_n(g_2)$.

We use the symbol $\backslash$ to denote removal of a member from a set. Thus $g \backslash n:m = \{i:j \mid i:j \in g, i:j \neq n:m\}$.

An allocation rule $Y: GR \to \mathbf{R}^N$ is *stable* iff:

$$\begin{gathered} \forall g \in GR, \forall n:m \in g, \\ Y_n(g) \geqslant Y_n(g \backslash n:m) \quad \text{and} \quad Y_m(g) \geqslant Y_m(g \backslash n:m). \end{gathered} \tag{5}$$

A stable allocation rule has the property that two players always benefit from reaching a bilateral agreement. So if the allocation rule were stable, then all players would want to be linked to as many others as possible, and we could expect the complete cooperation graph g^N to be the cooperation structure of the game.

In general, a characteristic function game can have many stable allocation rules. For example, consider the two-player "Divide the Dollar" game: $N = \{1, 2\}$, $v(\{1\}) = v(\{2\}) = 0$ and $v(\{1, 2\}) = 1$. To be an allocation rule for v, Y must satisfy $Y_1(\emptyset) = 0$, $Y_2(\emptyset) = 0$, and $Y_1(\{1:2\}) + Y_2(\{1:2\}) = 1$. ($\emptyset$ is the empty graph, with no links.) Stability then requires $Y_1(\{1:2\}) \geqslant 0$ and $Y_2(\{1:2\}) \geqslant 0$.

To narrow the range of allocation rules under consideration, we may seek allocation rules which are equitable in some sense. One equity principle we may apply is the equal-gains principle: that two players should gain equally from their bilateral agreement.

We define an allocation rule $Y : GR \to \mathbf{R}^N$ to be *fair* iff:

$$\forall g \in GR, \forall n : m \in g, \; Y_n(g) - Y_n(g \backslash n : m) = Y_m(g) - Y_m(g \backslash n : m). \qquad (6)$$

For example, in the Divide the Dollar game, the only fair allocation rule has $Y_1(\{1:2\}) = 0.5$ and $Y_2(\{1:2\}) = 0.5$, so that both players gain 0.5 units of transferable utility from their agreement link.

To state our main result, we need one more definition. Given a characteristic function game v and a graph g, define v/g to be a characteristic function game so that

$$\forall S \subseteq N, \qquad (v/g)(S) = \sum_{T \in S/g} v(T). \qquad (7)$$

(Recall the definition of S/g in (3).) One may interpret v/g as the characteristic function game which would result if we altered the situation represented by v, requiring that players can only communicate along links in g.

THEOREM. *Given a characteristic function game v, there is a unique fair allocation rule $Y : GR \to \mathbf{R}^N$(satisfying* (4) *and* (6)). *This fair allocation rule also satisfies*

$$Y(g) = \varphi(v/g), \quad \forall g \in GR,$$

where $\varphi(\cdot)$ is the Shapley value operator. Furthermore, if v is superadditive then the fair allocation rule is stable.

(Recall that a game v is *superadditive* iff: $\forall S \subseteq N, \forall T \subseteq N$, if $S \cap T = \emptyset$ then $v(S \cup T) \geqslant v(S) + v(T)$.)

(For proof, see §5.)

Since $v/g^N = v$ (where g^N is the complete graph on N), we get $Y(g^N) = \varphi(v)$ for the fair allocation rule Y. Thus our notions of cooperation graphs and fair allocation rules provide a new derivation of the Shapley value. (See Shapley [1953] and Harsanyi [1963] for other approaches.)

4. Example. Let $N = \{1, 2, 3\}$, and consider the characteristic function game v where:

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, \quad v(\{1,3\}) = v(\{2,3\}) = 6, \quad \text{and}$$
$$v(\{1,2\}) = v(\{1,2,3\}) = 12.$$

The fair allocation rule for this game is as follows:

$$Y(\emptyset) = (0,0,0), \qquad Y(\{1:2, 1:3\}) = (7,4,1),$$
$$Y(\{1:2\}) = (6,6,0), \qquad Y(\{1:2, 2:3\}) = (4,7,1),$$
$$Y(\{1:3\}) = (3,0,3), \qquad Y(\{1:3, 2:3\}) = (3,3,6),$$
$$Y(\{2:3\}) = (0,3,3), \qquad Y(\{1:2, 1:3, 2:3\}) = (5,5,2).$$

The Shapley value of v is $\varphi(v) = Y(g^N) = (5, 5, 2)$.

This example was chosen because most other well-known solution concepts, the core and the nucleolus and the bargaining set, all select the single allocation (6, 6, 0) for this game. These solution concepts are all based on ideas about what it means for the universal coalition N to be stable against objections. According to the argument for the core, (5, 5, 2) should be an unstable allocation because players 1 and 2 could earn 12 units wealth for themselves, which exceeds the wealth $5 + 5 = 10$ given to them. But when we shift our perspective from coalitions to cooperation graphs, this argument evaporates, and the value (5, 5, 2) actually is part of a stable fair allocation rule. If any one player were to break either or both of his cooperation links, then his fair allocation would decrease. To be sure, if both players 1 and 2 were to simultaneously break their links with 3, then both would benefit; but each would benefit even more if he continued to cooperate with player 3 while the other alone broke his link to player 3.

5. Proof of the theorem. We show first that there can be at most one fair allocation rule for a given game v. Indeed, suppose $Y^1 : GR \to \mathbf{R}^N$ and $Y^2 : GR \to \mathbf{R}^N$ both satisfy (4) and (6) and are different. Let g be a graph with a minimum number of links such that $Y^1(g) \neq Y^2(g)$; set $y^1 = Y^1(g)$ and $y^2 = Y^2(g)$, so that $y^1 \neq y^2$. By the minimality of g, if $n : m$ is any link of g, then $Y^1(g \backslash n : m) = Y^2(g \backslash n : m)$. Hence (6) yields

$$y_n^1 - y_m^1 = Y_n^1(g\backslash n : m) - Y_m^1(g\backslash n : m) = Y_n^2(g\backslash n : m) - Y_m^2(g\backslash n : m) = y_n^2 - y_m^2.$$

Transposing, we deduce

$$y_n^1 - y_n^2 = y_m^1 - y_m^2$$

whenever n and m are linked, and so also when they are in the same connected component S of g. Thus we may write $y_n^1 - y_n^2 = d_S(g)$, where $d_S(g)$ depends on S and g only, but not on n. But by (4) we have $\sum_{n \in S} y_n^1 = \sum_{n \in S} y_n^2$. Hence $0 = \sum_{n \in S} (y_n^1 - y_n^2) = |S| d_S(g)$, and so $d_S(g) = 0$. Hence $y^1 = y^2$ after all, a contradiction. That is, there can be at most one fair allocation rule for v.

It now remains only to show that $Y(g) = \varphi(v/g)$ implies that (4) and (6) are satisfied, along with (5) if v is superadditive.

We show (4) first. Select any $g \in GR$. For each $S \in N/g$, define u^S to be a characteristic function game such that:

$$u^S(T) = \sum_{R \in (T \cap S)/g} v(R), \quad \forall T \subseteq N.$$

Now, any two players connected in T by g are also connected in N by g, so

$$T/g = \bigcup_{S \in N/g} (T \cap S)/g.$$

Therefore $v/g = \sum_{S \in N/g} u^S$. But S is a carrier of u^S, because $u^S(T) = u^S(T \cap S)$. So, using the carrier axiom of Shapley [1953], for any $S \in N/g$ and any $T \in N/g$:

$$\sum_{n \in S} \varphi_n(u^T) = \begin{cases} u^T(N), & \text{if } S = T; \\ 0, & \text{if } S \cap T = \emptyset. \end{cases}$$

Thus, by linearity of φ, if $S \in N/g$ then

$$\sum_{n \in S} \varphi_n(v/g) = \sum_{T \in N/g} \sum_{n \in S} \varphi_n(u^T) = u^S(N) = \sum_{R \in S/g} v(R) = v(S).$$

To show (6) holds, select any $g \in GR$ and any $n : m \in g$. Let $w = v/g - v/(g\backslash n : m)$. Observe that $S/g = S/(g\backslash n : m)$ if $\{n, m\} \not\subseteq S$. So if $n \notin S$ or $m \notin S$ we get:

$$w(S) = \sum_{T \in S/g} v(T) - \sum_{T \in S/(g\backslash n : m)} v(T) = 0.$$

So the only coalitions with nonzero wealth in w are coalitions containing both n and m. So by the symmetry axiom of Shapley [1953] it follows that $\varphi_n(w) = \varphi_m(w)$. By linearity of φ, $\varphi_n(v/g) - \varphi_n(v/g\backslash n : m) = \varphi_n(w) = \varphi_m(w) = \varphi_m(v/g) - \varphi_m(v/g\backslash n : m)$.

Finally we show (5). Observe that $S/(g\backslash n : m)$ always refines S/g as a partition of S, and if $n \notin S$ then $S/(g\backslash n : m) = S/g$. So, if v is superadditive:

$$(v/g)(S) = \sum_{T \in S/g} v(T) \geqslant \sum_{T \in S/(g\backslash n : m)} v(T) = (v/(g\backslash n : m))(S)$$

and the inequality becomes an equality if $n \notin S$. Thus, if $w = v/g - v/(g\backslash n : m)$, then $w(S) \geqslant 0$ for all S and $w(S) = 0$ if $n \notin S$. Hence $w(S \cup \{n\}) \geqslant w(S)$ for all S, and so $\varphi_n(w) \geqslant 0$, by the representation of the Shapley value as an expected marginal contribution (see Shapley [1953]). Thus $\varphi_n(v/g) - \varphi_n(v/(g\backslash n : m)) = \varphi_n(w) \geqslant 0$, proving stability.

Acknowledgments. The author is indebted to Kenneth Arrow and Truman Bewley for many conversations on this subject, and to Robert Aumann for detailed and useful comments.

References

[1] Aumann, R. J. and Drèze, J. H. (1974). Cooperative Games with Coalition Structures. *International Journal of Game Theory* III 217–237.

[2] Harsanyi, J. C. (1963). A Simplified Bargaining Model for the n-Person Cooperative Game. *International Economic Review* IV 194–220.

[3] Shapley, L. S. (1953). A Value for n-Person Games. In *Contributions to the Theory of Games II*. H. W. Kuhn and A. W. Tucker, eds. Princeton, Princeton University Press. 307–317.

GRADUATE SCHOOL OF MANAGEMENT, NATHANIEL LEVERONE HALL, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201