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An axiomatic characterization of Bayes' Rule

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Abstract

This paper provides an axiomatic characterization of Bayes' Rule that is widely used for updating beliefs. Bayes' Rule is viewed as a revision rule. Consider an agent whose belief about a set of states is characterized by a point in a unit simplex of appropriate dimension. Now new information emerges that rules out the possible occurrence of some of the states. The revision rule then assigns new probabilities over the subset of states that is not ruled out. The paper provides a set of axioms that characterizes Bayes' Rule. The main axiom is Path Independence. A revision rule satisfies Path Independence if the probability distribution over any set of states is unaffected by the order in which new information comes in.

Keywords: Bayes' Rule; Revision rule; Path Independence

1. Introduction

Bayes' Rule is pervasive in theoretical economics, its widest use being for the purpose of updating beliefs. From the perspective of probability theory, Bayes' Rule can be derived as a consequence of the basic axioms of probability and the definition of conditional probability. This paper offers an alternative characterization of Bayes' Rule based on axioms inspired by those in the axiomatic theory of surplus sharing.

The central notion in the paper is that of a *revision rule*. Consider a situation where an agent has an initial or prior belief about the true state of the world. This belief is expressed in the form of a probability distribution over the set of "possible" states of the world, or geometrically by a point in the unit simplex of appropriate dimension. Now new

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information emerges which conclusively rules out the occurrence of certain states. A revision rule formulates an updated or posterior belief, which is a probability distribution over the states which remain "possible". It is clear that Bayes' Rule is a revision rule. In particular, it redistributes the aggregate probability weight of the states which are eliminated, among the states which remain, in proportion to the probability that is assigned to each of these remaining states by prior belief.

Revision Rules are also sometimes described in the literature as "Evidence based Rules". Evidence based rules appear in a wide variety of related contexts. In models of learning, Stahl (1996, 1999, mimeo) introduces a family of such evidence based rules in the context of learning dynamics. Belief revision rules are widely applied in other contexts as well, one prominent area being artificial intelligence or more specifically computer simulations of autonomous agents (Bhargava and Branley, 1995). Computer simulations form an important aspect of what is known as *decision support technology* and is widely used in formulating combat or military strategies. In such computer simulations, there are several schemes for representing meaningful information and various techniques for reasoning with information (Pearl, 1988; Sanchez and Zadeh, 1988). In such a computer-simulated world, at any instant, an agent has a previous belief (prior) and a set of information. The agent combines the set of information with the previous belief using some belief revision rule to obtain the current belief. Even generalizations of probability measures such as Dempster–Shafer type belief functions (Dempster, 1967; Shafer, 1976) use belief revision rules for combining ex ante uncertainty with current information. There are many ways to formulate belief revision rules, candidates being Bayesian methods and weighted combination of beliefs.

There is an extensive literature on various ways of characterizing Bayes' Rule. Most of such methods are from a *no-arbitrage* perspective. No arbitrage is a fundamental principal of economic rationality. The arbitrage principal has a long history. In the literature on Bayesian Statistics and decision theory, it was introduced as an axiom by de Finetti (1974), for characterizing subjective probability. More recently the "arbitrage principle" has been proposed as a foundation for noncooperative game theory through its dual relation with the concept of correlated equilibrium (McCardle and Nau, 1990; Nau, 1992). McCardle and Nau (1991) tries to unify decision theory, market theory and game theory by appealing to the principle of no arbitrage. However, in all these settings money plays a crucial role as a medium of communication. In environments where money is available as a medium of exchange and measurement, no arbitrage is synonymous with subjective utility maximization in personal decisions. The point of difference in characterizations involving the no-arbitrage principle and the one in this paper is that this paper tries to axiomatize Bayes' Rule without introducing money in the model.

The main result of the paper is a characterization of Bayes' rule in terms of axioms imposed on revision rules. The most potent of these axioms is *Path Independence*, an axiom which has been employed in a variety of contexts such as the theory of rational choice (Plott, 1977, axiomatic bargaining (Kalai, 1977), etc). The axiom requires that the posterior belief be unaffected by the order in which the new information appears. In Section 3, this axiom is illustrated by means of an example. The other axioms in the characterization are relatively innocuous. One is a symmetry (or anonymity) axiom, which requires that the names of the states of the world are not material for the revision rule. The continuity axiom requires the revision rule to be continuous with respect to the prior. The monotonicity axiom requires that the revised probability on a state should not be less than the prior on that state. Finally a "no mistake hypothesis" is imposed which requires that if an agent believes initially that the occurrence of a particular state is impossible, then she continues to believe this even after the arrival of new information. (Actually this axiom is required only in the very special case where a revision eliminates all but only two states of the world.)

A paper, which is related in spirit to the present one, is Rubinstein and Zhou (1999). They consider a general decision situation where an agent chooses an element from a set S given a reference point e. The set S is a suitable subset of an ambient space X. For the case of updating beliefs, X can be the set that includes all possible theories (point beliefs) about the world. Assuming S to be a convex subset of an Euclidean space, they axiomatize the choice rule that selects a point in S that is closest to e. Their paper uses a strong symmetry axiom that forces choice decisions along the line joining the minimum distance point and e. The present paper considers choices on unit simplices and characterizes a different rule.

In terms of structure, the problem analyzed in this paper is similar to the so-called "bargaining problem with claims" (Chun and Thomson, 1992). That problem has the structure of a triple (S, e, c) with the interpretation that S is the set of feasible utility vectors, $e \in S$ is the disagreement point and $c \notin S$ is the vector of claims that cannot be fulfilled. In such a setting, Chun and Thomson characterize the proportional solution, which is similar in functional form to the Bayes' Rule. That model however emphasizes the utility interpretation of choices and as a consequence Pareto optimality is imposed as an axiom. The present model however does not have a utility interpretation and so Pareto optimality is not imposed.

The structure of the problem is also closely related to the one used in the analysis of bankruptcy problems (see O'Neill, 1982; Aumann and Maschler, 1985). The issue there is to divide the liquidation value of a bankrupt firm among its creditors. In this context, Chun (1988) characterizes the proportional solution, which is again Bayes' Rule. However, that model uses a strong axiom the *No-Advantageous Reallocation* (NAR) (for a discussion of NAR, see Moulin, 1987), which is a stronger version of the Pareto optimality criterion.

The paper is organized as follows: in Sections 2 and 3, we give the model and the axioms. Section 4 gives the main result, while Section 5 checks the tightness of the axiomatic characterization.

2. Model

Let $T=\{1, \ldots, t\}$ denote the finite set of states of the world. Let $\mathscr{P}(T)$ denote the class of all nonempty subsets of T. Generic elements of $\mathscr{P}(T)$ are denoted by P, Q, R, etc. For any $P \in \mathscr{P}(T)$ define $\Delta^P = \mathrm{conv} - \mathrm{hull}\{e^i\}_{i \in P}$ where e^i is a vector in $\Re^P$ for which the ith coordinate is 1 and the rest are zeros. Thus Δ^P is the $|P| - 1$ dimensional simplex.

Before proceeding further some preliminary definitions are needed.

Definition 1 (Revision Rule). Consider any $Q \in \mathscr{P}(T)$ and $x \in \Delta^Q$. Consider any $P \subset Q$ such that there is at least one $j \in P$ for which $x_j > 0$. A revision rule $F(P, Q, x)$ is a function that assigns a unique point $F(P, Q, x) \in \Delta^P$ with the restriction $F(Q, Q, x) = x$.

Now $F(P, Q, x)$ is a $|P|$ dimensional vector. The ith element is $F_i(P, Q, x)$. Thus

$$F(P,Q,x) = (F_i(P,Q,x))_{i\in P} \tag{1}$$

Definition 2 (Bayes' Rule). Consider $Q\in\mathscr{P}(T)$, $x\in\Delta^Q$. Consider any $P\subset Q$ such that there exists at least one $i\in P$ for which $x_i>0$. Then Bayes' Rule BR(P, Q, x) is the revision rule having the following expression: $\forall i\in P$,

$$\mathrm{BR}_i(P,Q,x) = x_i + \left(\sum_{j\in Qd\backslash P} x_j\right)\frac{x_i}{\sum_{k\in P} x_k} \tag{2}$$

3. Axioms

We would like to characterize Bayes' Rule. To that end we consider the following axioms.

3.1. Path independence (PI)

Consider P, Q, $R\in\mathscr{P}(T)$, $P\subset Q\subset R$ and $x\in\Delta^R$. A revision rule satisfies PI if and only if

$$F(P,Q,F(Q,R,x)) = F(P,R,x) \tag{3}$$

The expression in Eq. (3) can alternatively be written in the following way: consider $P\in\mathscr{P}(T)$ and take Q_1, $Q_2\in\mathscr{P}(T)$ such that $Q_1\supset P$ and $Q_2\supset P$. PI then says,

$$F(P,T,x) = F(P,Q_1,F(Q_1,T,x)) = F(P,Q_2,F(Q_2T,x)) \tag{4}$$

Path Independence is a consistency requirement. Path Independence implies that the order in which information comes in does not matter. The axiom is illustrated by the following example. Suppose that a person running a high fever consults a doctor. Initial symptoms suggest to the doctor that the true disease is one in the set $\{D_1, D_2, D_3, D_4, D_5\}$. His beliefs are represented by a probability distribution over this set. The doctor orders blood test B_1 which can correctly identify D_4 and D_5 and blood test B_2 which can correctly identify D_3. Both the tests are negative. The doctor's revision rule transforms his prior beliefs into a probability distribution over $\{D_1, D_2\}$. Suppose the results on B_1 arrive before that on B_2. The posterior on $\{D_1, D_2\}$ can be thought of as passing through an intermediate belief on $\{D_1, D_2, D_3\}$. If on the other hand the report on B_2 precedes that on B_1, the prior is first revised to $\{D_1, D_2, D_4, D_5\}$ and eventually to $\{D_1, D_2\}$. If a revision rule satisfies path independence, the same posterior (on $\{D_1, D_2\}$) obtains in both the cases.

3.2. Symmetry (SYM)

Consider any P, $Q \in \mathscr{P}(T)$, $P \subset Q$ and $x \in \Delta^Q$. Consider any permutation function σ: $Q \to Q$ such that

[i] $\sigma(i) \in P$ if $i \in P$

[ii] $\sigma(i) = i\ \forall i \notin P$

A revision rule satisfies SYM if and only if

$$\forall i \in P, \quad F_{\sigma(i)}(P, Q, \sigma(x)) = F_i(P, Q, x) \quad \text{where,}\ \ \sigma(x) = (x_{\sigma(k)})_{k \in Q} \tag{5}$$

This is an anonymity requirement. It forces the revision rule to ignore the names of the states of the world. In the disease example, the doctor should not be putting more weight on a disease just because it carries a particular name, say tuberculosis.

3.3. Continuity (CONT)

The continuity axiom requires that the revision rule $F(P, Q, x)$ is continuous in x.

This axiom means a small change in the prior belief should not lead to any abrupt jump in the revised probabilities.

3.4. Monotonicity (MON)

Consider any P, $Q \in \mathscr{P}(T)$, $P \subset Q$ and $x \in \Delta^Q$. A revision rule satisfies monotonicity if for all $i \in P$,

$$F_i(P, Q, x) \geq x_i \tag{6}$$

This monotonicity requirement says that, if a state is not ruled out by some new information coming in, then the revised probability on that state is not going to be less than the prior probability.

3.5. No mistake hypothesis (NM)

For all $P \in \mathscr{P}(T)$ with $|P| = 2$, if $x_i = 0$ for some $i \in P$, then

$$F_j(P, T, x) = 1, \qquad j \in P, j \neq i \tag{7}$$

Let us consider the disease example again. Suppose that the prior belief of the doctor about disease D_1 is zero. This axiom says that if the doctor believes that it is impossible for disease D_1 to occur and if the test conducted does not rule out D_1, then, after the revision process, the doctor is never going to put positive weight on D_1. The agent is therefore not allowed to make mistakes of a particular kind.

4. The main result

Let F be a revision rule and let $x \in \Delta^Q$, $P \subset Q \subset T$. Without loss of generality, we can write

$$F_i(P, Q, x) = x_i + \phi_i^{P,Q}(x), \quad \forall i \in P \tag{8}$$

where $\phi_i^{P,Q}$: $\Delta^Q \rightarrow \Re$ is a real valued function with the restriction $-x_i \leq \phi_i^{P,Q}(x) \leq 1 - x_i$, for any $x \in \Delta^Q$. Since for any $Q \in \mathscr{P}(T)$, $x \in \Delta^Q$ necessarily means $x \in \Delta^T$, we ignore the second superscript.

Theorem 1. *Suppose* $|T| = 3$*. A revision rule satisfies SYM, CONT, MON and NM if and only if it is the Bayes' Rule.*

Without loss of generality we can take $T = \{1, 2, 3\}$. Before going into the proof of the theorem let us consider the following lemma.

Lemma 1. *Let* $P = \{1, 2\}$, $T = \{1, 2, 3\}$*. A revision rule satisfies SYM, CONT and MON if and only if there exists a continuous function* $g: \Re^2_+ \rightarrow \Re$ *such that* $\forall i \in P$*, and for all* $x \in \Delta^T$*,*

$$F_i(P, T, x) = x_i + \frac{x_i}{x_l + x_2}\{x_3 - 2g(x_1 + x_2, x_3)\} + g(x_1 + x_2, x_3)$$

Proof. As mentioned above, for each i in P, the revision rule can be written as

$$F_i(P, T, x) = x_i + \phi_i^P(x_1, x_2, x_3) \tag{9}$$

where x_1 is the first element of the vector, x_2 is the second element and so on. Using MON we can say that $\phi_i^P(x_1, x_2, x_3) \geq 0$. Now consider σ: $\{1, 2, 3\} \rightarrow \{1, 2, 3\}$ in the following manner: $\sigma(1) = 2$, $\sigma(2) = 1$, $\sigma(3) = 3$. Then from SYM it follows,

$$F_2(P, T, x) = F_1(P, T, \sigma(x)) = x_2 + \phi_1^P(x_2, x_1, x_3)$$

Therefore, $\phi_2^P(x_1, x_2, x_3) = \phi_1^P(x_2, x_1, x_3)$. Since $F_1(P, T, x) + F_2(P, T, x) = 1$, it follows that

$$\phi_1^P(x_1, x_2, x_3) + \phi_1^P(x_1, x_2, x_3) = x_3 \tag{10}$$

Since $(x_1+x_2,\ 0,\ x_3)\in\Delta^T$, it also follows that

$$\phi_1^P(x_1+x_2,0,x_3)+\phi_1^P(0,x_1+x_2,x_3)=x_3 \tag{11}$$

Combining Eqs. (10) and (11), we have

$$\phi_1^P(x_1,x_2,x_3)+\phi_1^P(x_2,x_1,x_3)=\phi_1^P(x_1+x_2,0,x_3)+\phi_1^P(0,x_1+x_2,x_3) \tag{12}$$

Define the function f: $\Re^3\rightarrow\Re$ as follows:
Let $z=(z_1,\ z_2,\ z_3)\in\Re^3$,

$$f(z_1,z_2,z_3)=\phi_1^P(z_1,z_2-z_1,z_3)-\phi_1^P(0,z_2,z_3).$$

Then, $f(x_1,\ x_1+x_2,\ x_3)+f(x_2,\ x_1+x_2,\ x_3)$

$$\begin{aligned}&=\phi_1^P(x_1,x_2,x_3)+\phi_1^P(x_2,x_1,x_3)-2\phi_1^P(0,x_1+x_2,x_3)\\&=\phi_1^P(x_1+x_2,0,x_3)-\phi_1^P(0,x_1+x_2,x_3)\\&=f(x_1+x_2,x_1+x_2,x_3)\end{aligned} \tag{13}$$

Thus f is additive with respect to the first argument for each (x_1+x_2), x_3. Since f is continuous (follows from CONT), applying the theorem on Cauchy Equation to Eq. (13), [1] it follows that there exists a function h: $\Re^2\rightarrow\Re$ such that,

$$f(x_i,x_1+x_2,x_3)=x_ih(x_1+x_2,x_3) \tag{14}$$

Since $\phi_1^P(x_1,\ x_2,\ x_3)-\phi_1^P(0,\ x_1+x_2,\ x_3)=f(x_1,\ x_1+x_2,\ x_3)$, we have,

$$x_1h(x_1+x_2,x_3)+\phi_1^P(0,x_1+x_2,x_3)=\phi_1^P(x_1,x_2,x_3) \tag{15}$$

Similarly,

$$x_2h(x_1+x_2,x_3)+\phi_1^P(0,x_1+x_2,x_3)=\phi_1^P(x_2,x_1,x_3) \tag{16}$$

Adding Eqs. (14) and (15) and using Eq. (10), we obtain

$$\begin{aligned}&(x_1+x_2)h(x_1+x_2,x_3)+2\phi_1^P(0,x_1+x_2,x_3)=x_3\\&\quad\Rightarrow h(x_1+x_2,x_3)=\frac{1}{x_1+x_2}\{x_3-2\phi_1^P(0,x_1+x_2,x_3)\}\end{aligned} \tag{17}$$

[1] For a treatment of Cauchy Equations, see Eichhorn (1978).

Substituting Eq. (17) in Eqs. (14) and (15) we obtain,

$$F_i(P,T,x) = x_i + \frac{x_i}{x_1+x_2}\{x_3 - 2\phi_1^P(0,x_1+x_2,x_3)\} + \phi_1^P(0,x_1+x_2,x_3) \quad i=1,2. \tag{18}$$

Writing the function $\phi_1^P(0, x_1+x_2, x_3)$ as the function g: $\Re_+^2 \to \Re$, we obtain the desired conclusion. □

Proof of Theorem 1. Without loss of generality let P={1, 2} and T={1, 2, 3} and $x\in\Delta^T$. Let $x_1=0$ be given. Then by NM we have, $F_1(P, T, x)=0$. Now from the definition of g given in Eq. (17) this implies that $g(x_1+x_2, x_3)=0$. Observe that $g(,)$ is the same for all $i\in P$. Thus we have $F_i(P, T, x)=(x_i)/(x_1+x_2)$ for all $i\in P$. Suppose now that $x\in\Delta^T$ and $x_i>0$ $\forall_i\in T$. Consider another vector $y\in\Delta^T$ defined as follows:

$$y_1 = 0, y_2 = x_1 + x_2, y_3 = x_3$$

Observe that g is the same for both x and y. But $g(y_1+y_2, y_3)=0$. This implies $F_i(P, T, x)=(x_i)/(x_1+x_2)$ for all $i\in P$ as desired. □

Now we consider the more general case.

Theorem 2. *Suppose $|T|\geq 4$. Then a revision rule satisfies SYM, PI, CONT and NM if and only if it is Bayes' Rule.*

The proof of the theorem follows from the given lemma.

Lemma 2. *Consider T such that $|T| = t \geq 4$ and $x\in\Delta^T$. If $F(P, T, x) = BR(P, T, x)$ for all $P\in\mathscr{P}(T)$ such that $|P| = m$ $(2\leq m< t)$, then $F(Q, T, x) = BR(Q, T, x)$ for all Q such that $|Q|>m$.*

Proof. The following cases are considered.

Case A: consider $x\in\Delta^T$ such that $x_k>0$ for all $k\in T$. Consider P, $Q\in\mathscr{P}(T)$ such that $Q=P\cup\{j'\}$, $j'\in T\setminus P$.

Fix an $i\in P$. From PI we get,

$$F_i(P,T,x) = F_i(P,Q,F(Q,T,x)) \Rightarrow \phi_i^Q(x) + \phi_i^P((x_j+\phi_j^Q(x))_{j\epsilon Q}) = \phi_i^P(x)$$

$$\Rightarrow \phi_i^Q(x) + (x_{j'}+\phi_{j'}^Q(x))\frac{x_i+\phi_i^Q(x)}{\sum\limits_{k\in P} x_k+\phi_k^Q(x)} \tag{19}$$

$$= \left(\sum_{m\notin P} x_m\right)\frac{x_i}{\sum\limits_{k\in P} x_k} \tag{20}$$

The last equality follows from the fact that $F(P, T, x) = \mathrm{BR}(P, T, x)$. Let $x_{j'} + \phi_{j'}^{Q}(x) = A$. Then we get,

$$\frac{\phi_i^Q(x) + Ax_i}{1 - A} = \left(1 - \sum_{k \in P} x_k\right) \frac{x_i}{\sum_{k \in P} x_k} \tag{21}$$

The last equality holds for any $j \in P$. So for any $j \in P$ we get,

$$\phi_j^Q(x) = \frac{x_j}{x_i} \phi_i^Q(x) \tag{22}$$

Now consider a P' in which a $j \in P \backslash i$ is replaced by state of the world j'. Thus $P' = (P \backslash j) \cup \{j'\}$. And one gets,

$$\phi_{j'}^Q(x) = \frac{x_{j'}}{x_i} \phi_i^Q(x) \tag{23}$$

Now, $\sum_{l \notin Q} x_1 = \sum_{j \epsilon Q} \phi_j^Q(x)$ This implies that for any $i \in Q$,

$$\phi_i^Q(x) = \left(\sum_{l \notin Q} x_i\right) \frac{x_i}{\sum_{j \in Q} x_j} \tag{24}$$

Case B: suppose that $x_k = 0$ for some $k \in T$. Consider a P with $|P| = m$ such that $k \in P$. Consider $Q \supset P$ such that $|Q| = |P| + 1$. Proceeding as above one can show that

$$\phi_k^Q(x) + A\frac{\phi_k^Q(x)}{1 - A} = 0 \Rightarrow \phi_k^Q(x) = 0$$

For any other $i \in Q$ such that $x_i > 0$ application of Case A gives

$$\phi_i^Q(x) = \left(\sum_{l \notin Q} x_l\right) \frac{x_i}{\sum_{j \in Q} x_j} \tag{25}$$

Thus we have seen that given $F(P, T, x) = \mathrm{BR}(P, T, x)$, for any P with $|P| = m$, $F(Q, T, x) = \mathrm{BR}(Q, T, x)$ whenever $Q = P \cup \{j\}$ for any $j \in T \backslash P$. Suppose that $F(Q, T, x) = \mathrm{BR}(Q, T, x)$ for any Q such that $m < |Q| \leq n < t$. Consider $Q' = Q \cup \{j'\}$ where $j' \in T \backslash Q$. Applying the procedure used above we can show that $F(Q', T, x) = \mathrm{BR}(Q', T, x)$. Therefore, we have the desired result. □

Proof of Theorem 2. Consider $x \in \Delta^T$ and $P \in \mathscr{P}(T)$. Now take $P' \subset Q \subset P$, such that $|P'| = 2$, $|Q| = 3$. Let $F(Q, T, x) = y$. Now $y \in \Delta^Q$. From Theorem 1, we get $F(P', Q, y) = \mathrm{BR}(P', Q, y)$. Now from PI we get, $F(P', Q, y) = F(P', Q, F(Q, T, x)) = F(P', T, x)$. So, $F(P', T, x) = \mathrm{BR}(P', T, x)$. Now from Lemma 2 we know that if $F(P', T, x) = \mathrm{BR}(P', T, x)$, then for any $Q \supset P'$, $F(Q, T, x) = \mathrm{BR}(Q, T, x)$. Since $P \supset P'$ we have $F(P, T, x) = \mathrm{BR}(P, T, x)$. □

Remark 1. There is a possible extension to the model considered above. Observe that the revision process analyzed in this paper always takes place from one set to its subsets. A possible way to extend this model would be to consider revisions that takes place from one set to another which is not necessarily a subset of the former. For the revision process to be meaningful the two sets should have nonempty intersection. Consider for example P, $Q \in \mathscr{P}(T)$, $P \cap Q \neq \emptyset$ and $x \in \Delta^T$. The choice rule for any such P, Q would be defined as $F(P, T, x) \in \Delta^P$ with the additional restrictions:

$$F(P, Q, F(Q, T, x)) \in \Delta^{P \cap Q} \tag{26}$$

In this extension let us consider an alternative version of the path independence axiom, which is due to Rubinstein and Zhou (1999).

[v] PI*: consider P, $Q \in \mathscr{P}(T)$ $x \in \Delta^T$, $P \cap Q \neq \emptyset$. Then,

$$F(P, Q, F(Q, T, x)) = F(P \cap Q, T, x) \tag{27}$$

Let $T = \{1, 2, 3\}$. Let $P = \{1, 2\}$, $Q = \{2, 3\}$. From PI* we get $F_2(P, Q, F(Q, T, x)) = F_2(\{2\}, T, x) = 1$. This implies $F_1(\{1, 2\}, \{2, 3\}, F(\{2, 3\}, T, x)) = 0$. Let $F(\{2, 3\}, T, x) = y$. Now $y_1 = 0$, i.e., $F_1(\{1, 2\}, \{2, 3\}, (0, y_2, y_3)) = 0$. Applying this to the expression in Eq. (17) we get $g(,) = 0$. Hence, $F(P, T, x) = \mathrm{BR}(P, T, x)$. For T with $|T| \geq 4$ the result follows from Lemma 2.

Thus we get an alternative characterization.

Theorem 3. *A choice rule satisfies SYM, MON, PI* and CONT if and only if it is the Bayes' Rule.*

Below we show that the four axioms are independent. For each axiom we give an example of a function that satisfies the remaining three but fails to satisfy it.

5. Independence of the axioms

1. *Example of a function that satisfies PI, MON, SYM and CONT but not NM.*
Let T={1, 2, 3, 4}. For any $R \in \mathscr{P}(T)$, define $F(R, T, x)$ as follows:

$$F_i(R,T,x) = 1/r \forall i \in R \quad \text{where } r = |R|.$$

Consider P={1, 2}, $x \in \Delta^T$, x=(0, α, β, γ,) where α, β, $\gamma \in (0, 1)$. This function satisfies PI, MON, CONT and SYM but not NM as $F_1(\{1, 2\}, T, x) = 1/2 \neq 0$.
2. *Example of a function that satisfies PI, MON, SYM and NM but not CONT.*
Again take T={1, 2, 3, 4}. For any $R \in \mathscr{P}(T)$, define $F(R, x)$ as follows:

$$F_i(R,T,x) = 1/m \quad \text{if } \ x_i > 0 = 0 \ \text{ otherwise.}$$

$$\text{where } m = |\{j \in M | x_j > 0\}|.$$

Take P={1, 2, 3}, $x \in \Delta^T$, x=(0, α, β, γ) where α, β, $\gamma \in (0, 1)$. Consider x_ϵ=(3ϵ, $\alpha-\epsilon$, $\beta-\epsilon$, $\gamma-\epsilon$); $F_1(\{1, 2, 3\}, T, x_\epsilon) = 1/3$ but $F_1(\{1, 2, 3\}, T, x) = 0$.
3. *Example of a function that satisfies NM, MON, SYM and CONT but not PI.*
Take T={1, 2, 3, 4}. Define,

$$F_i(P,T,x) = \frac{x_i}{\sum_{k \in P} x_k} \quad \text{if } \ |P| = 2.$$

$$= 1/p \ \ (\text{where } \ p = |P|) \ \text{ otherwise.}$$

Consider P={1, 2}, Q={1, 2, 3}. Consider $x \in \Delta^T$ such that x=(0.1, 0.2, 0.3, 0.4). This function satisfies NM, MON, CONT, SYM but not PI.
4. *Example of a function that satisfies NM, PI, CONT, SYM but not MON.*
Take T={1, 2, 3, 4}. Define the revision rule as follows: If $R \in \mathscr{P}(T)$ and $|R|$=2,

$$F_i(R,T,x) = \frac{x_{i^2}}{\sum_{k \in R} x_{k^2}}$$

Otherwise,

$$F_i(R,T,x) = \frac{x_1}{\sum_{k \in R} x_k}$$

Consider $x \in \Delta^T$ such that $x=(0.05,\ 0.85,\ 0.025,\ 0.075)$. Take $R=\{1,\ 2\}$. Then $F_1(R, x)=0.0034<0.05$.

5. *Example of a function that satisfies NM, PI, MON and CONT but not SYM.*

Take $T=\{1, 2, 3, 4\}$. For any $R \in \mathscr{P}(T)$ define F as follows: If $|R|=2$

$$F_i(R, T, x) = \frac{2x_i}{2x_i + x_j} \text{ if } i = \max\{k \,|\, k \in R\}$$

$$= \frac{x_i}{x_1 + 2x_j} \text{ otherwise.}$$

Otherwise,

$$F_i(R, T, x) = \frac{x_i}{\sum_{k \in R} x_k}$$

Consider $P=\{1, 2\}$ and $\sigma\{1, 2, 3, 4\}$ as follows: $\sigma(1)=2$; $\sigma(2)=1$; $\sigma(3)=3$; $\sigma(4)=4$. Then $F_{\sigma(2)}(\{1,\ 2\},\ T,\ \sigma(x))=x_2/(x_2+2x_1)$ but $F_2(\{1,\ 2\},\ T,\ x)=2x_2/(x_1+2x_2)$. This function satisfies PI, NM, MON, CONT but not SYM.

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