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About inheritance distribution

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Abstract

In this work, we propose two axiomatic procedures to distribute an inheritance based on monetary compensations. In a cooperative context, the differences among the agents' evaluations of the goods are used to increase everyone's participation. When the goods' valuations are heterogeneous each agent gets much more than $1/n$ of the value he believes the inheritance has.

The two procedures are characterized by three criteria, namely: money preservation, Pareto optimality, and a certain kind of proportionality. The difference of the procedures is determined by the kind of proportionality used.

Keywords: Inheritance; Axiomatic procedure; Pareto optimality

1. Introduction

In this work, we present two axiomatic procedures in order to distribute an inheritance. The first procedure is uniquely determined by three properties: preservation of money (implicit in the definition of the solution), Pareto optimality and being in proportion with the agents' monetary valuations of items. The second one is uniquely determined by almost the same three properties, the last one is changed by being in proportion with the agents' monetary valuations of the inheritance. In both cases, after each agent evaluates each item, the heterogeneity of the agents' preferences is used to increase everybody's participation.

Both procedures have several properties in common: let n be the number of agents, then every agent gets at least $1/n$ of what he believes is the value of the inheritance and this fraction increases when the items evaluations are heterogeneous. The solutions are monotonic with respect to the items. Another pair of common properties are that a change in the scale of items' evaluation yields an equivalent change in the total amounts agents receive and also, the solutions are continuous functions of the item's evaluations.

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The first procedure could be applied in steps: first some items and later others, and money could be distributed "in" or "out" of the procedure with the same result. We can think of this procedure as if every item is assigned to the highest bidder and the collected money is redistributed in proportion to the bids. The second procedure does not have the previous properties, but with their solutions every agent gets the same percentage of the value he believes is the whole value of the inheritance and this percentage is the highest that can be guaranteed for all the agents. Besides, when a new heir is included, keeping the same inheritance, the agents already present all benefit or all suffer. In this case, the collected money is redistributed in such a way that a common percent of the inheritance is as big as possible, each one in his corresponding evaluation.

The axiomatization of both procedures are also generalized in the case where the agents have different rights over the inheritance in a simple and natural way. Furthermore, all procedures are easy to use in real life and the solutions change in a smooth way as a function of the matrix A.

Both procedures are similar to the procedure of Knaster–Steinhaus (see, for example, Brams and Taylor (1996), p. 68). This procedure assign the items by an auction, gives to the agents an initial fair share ($1/n$ of the value he believes the items has) and then assign equal share of the surplus. Raith (2000) formulates three similar two-step fair division algorithms for negotiations: in the first step ensures an Pareto outcome, and in the second step he establishes 'fairness' through a distribution of gains. Tadenuma and Thomson (1993) consider the problem of fairly allocating an indivisible good when monetary compensations to the agents who do not receive the good are possible. The idea of assigning objects to the highest bidders was present in Steinhaus (1946). Adjusted Winner procedure of Brams and Taylor (1996, p. 68) allocates a set of divisible's goods among two agents without using money.

In Section 2 we provide a formal characterization of both procedures based on several definitions that describe individual features of the procedures. The main results are given in Theorems 1 and 3, with two further variants characterized by Theorems 2 and 4. In Section 3, we list further properties of the procedures and ends with a brief discussion of the results. All mathematical proofs are provided in the Appendix A.

Before we start, we want to clarify the framework of this work. We proceed, according to what is used in cooperative games by using any disagreement about the item's values to improve everyone's participation. Our main goal is to establish normative solutions based on desirable properties. Another option is that each agent tries to take advantage of any knowledge he has over the preferences of the others. This situation can be modeled as a non-cooperative game, see for example, Thomson (1988) or Tadenuma and Thomson (1993), but we leave this approach for future work.

2. Axiomatic characterization

By an Inheritance's Distribution problem (hereafter a "problem") we mean a triplet (M, N, A), where $N = \{1, \dots, n\}$ is a finite set of agents that inherits a finite set $M = \{1, \dots, m\}$ of goods and A is a $m \times n$ positive matrix, where a_{ij} entry is the value that agent j gives to the item i. In order to generate this matrix, a judge can ask the agents to give a bid for each item; each agent independently from the others.

Now, we turn our attention to the solution concept. First, we propose monetary compensations for an arbitrary assignment of the goods, and then we use the Pareto optimality property to get an assignment of the goods. Suppose for a moment that the goods are already assigned, i.e. suppose a partition $\{M_1, \ldots, M_n\}$ of M is given, where M_j is the set of goods agent j has received and let us denote by $u_j = \sum_{i \in M_j} a_{ij}$, the value that agent j gives to it. Some of these sets could be empty (in such a case, the corresponding entry for u is defined to be zero). Let $x \in \mathbb{R}^N$ be a vector of the amounts of money agents receive as compensation corresponding to u. So, the agent j receives x_j in money (or he pays, in case x_j is negative) and the set M_j of goods that he evaluated in u_j units of money.

Definition 1. We will say that $u \in \mathbb{R}^N$ is the result of an assignment of the goods in M, if there exists a partition $\{M_1, \ldots, M_n\}$ of M such that $u_j = \sum_{i \in M_j} a_{ij}$ for every $j \in N$.

Definition 2. We will say that u came from the highest bidders if u is the result of an assignment of the goods in M with a partition $\{M_1, \ldots, M_n\}$ of M such that if $k \in M_j$ then $a_{kj} \geq a_{ki}$ for every $i \in N$.

Definition 3. By a solution of (M, N, A), we mean a pair (u, x) where u is the result of an assignment of the goods in M and $x \in \mathbb{R}^N$ is a pay-off vector whose coordinates add up to zero, i.e. $\sum_{j \in N} x_j = 0$.

Now, we establish the properties that characterize the first procedure. One of the properties we will use to determinate our procedures is Pareto optimality. A solution is Pareto optimal if there is no other solution that will make one or more agents better off leaving the others as well off as they were. The next definition gives shape to this idea.

Definition 4. A solution (u, x) is Pareto optimal if and only if there does not exist any other solution $(\tilde{u}, \tilde{x})$ such that

$$\tilde{u} + \tilde{x} \geq u + x$$

with strict inequality holding true for at least one coordinate.

Lemma 1. *In the solution (u, x), u came from the highest bidders if and only if (u, x) is Pareto optimal.*

Proof. Let (u, x) be a solution where u came from the highest bidders and suppose there exists a solution $(\tilde{u}, \tilde{x})$ such that $\tilde{u} + \tilde{x} \geq u + x$ with strict inequality for at least one coordinate. Then, adding the n inequalities we get $\sum_{j \in N} \tilde{u}_j > \sum_{j \in N} u_j$ because $\sum_{j \in N} x_j = \sum_{j \in N} \tilde{x}_j = 0$. But, as u came from the highest bidders, $\sum_{j \in N} \tilde{u}_j \leq \sum_{j \in N} u_j$,which is a contradiction. So (u, x) is Pareto optimal.

For the converse, let (u, x) be a Pareto optimal solution and assume u does not come from the highest bidders. Then, there exists a good assigned to an agent who is not the highest bidder. In order to get a solution where two agents improve, the corresponding agent could sell this good to any agent who gives it a higher value with a price between their valuations. This contradicts the Pareto optimality of (u, x), so u comes from the highest bidders. □

For any $P \subseteq M$, let us denote by A_P the matrix A restricted to the rows in P. Consider the set of problems $\{(P, N, A_P)|P \subseteq M\}$ and a corresponding family of solutions $\{(u(P), x(P))|P \subseteq M\}$ for (M, N, A). For a given solution $(u(P), x(P))$ of the problem (P, N, A_P) where $P \subseteq M$, we will denote by $\varphi(P) = u(P) + x(P)$ the total value agents receive. In case we need to specify the set of agents, we will include it as a second coordinate, i.e. $\varphi(P, N) = u(P, N) + x(P, N)$.

The goods have no value by themselves. This value is generated by the agents' evaluations and so, we feel that it is according to them that it must be distributed. So, we suppose that the changes in the value agents receive, when one item is omitted, are in proportion with the value they give to this item.

Definition 5. We will say that a family $\{(u(P), x(P))|P \subseteq M\}$ of solutions for (M, N, A) is in proportion with A if and only if

$$(\varphi_i(P) - \varphi_i(P\backslash\{k\}))a_{kj} = (\varphi_j(P) - \varphi_j(P\backslash\{k\}))a_{ki} \tag{2.1}$$

for every $i, j \in N$ and $k \in P$.

Lemma 2. *Let $\{(u(P), x(P))|P \subseteq M\}$ be a family of solutions for (M, N, A) in proportion with A and $m(P) = \sum_{j\in N} \varphi_j(P)$ then*

(a) $\varphi_i(P) = \varphi_i(P\backslash\{k\}) + [m(P) - m(P\backslash\{k\})]a_{ki}/\sum_{j\in N} a_{kj}$, *for every $P \subseteq M$ and $k \in P$.*
(b) $m(P) = \sum_{k\in P} m(\{k\})$.
(c) *Moreover, if the family is Pareto optimal then $m(\{k\}) = \max_{i\in N} a_{ki}$ for every $k \in M$.*

Proof.

(a) Consider a family $\{(u(P), x(P))|P \subseteq M\}$ of solutions for (M, N, A), an arbitrary subset $P \subseteq M$ and $k \in P$. Adding the corresponding equalities in (2.1) over $j \in N$, we get

$$\begin{aligned}(\varphi_i(P) - \varphi_i(P\backslash\{k\})) \sum_{j\in N} a_{kj} &= \sum_{j\in N} (\varphi_j(P) - \varphi_j(P\backslash\{k\}))a_{ki} \\ &= [m(P) - m(P\backslash\{k\})]a_{ki},\end{aligned}$$

thus

$$\varphi_i(P) = \varphi_i(P\backslash\{k\}) + [m(P) - m(P\backslash\{k\})]\frac{a_{ki}}{\sum_{j\in N} a_{kj}}. \tag{2.2}$$

(b) After adding (2.2) over $i \in N$, the proof is direct by induction on the cardinality of P using the facts that $m(\emptyset) = 0$ and $\varphi_i(\emptyset) = 0$ for all $i \in N$.

(c) By (b), we have

$$m(\{k\}) = \sum_{i\in N} \varphi_i(P) = \sum_{i\in N} u_i(P) + x_i(P) = \sum_{i\in N} u_i(P) = a_{kj}.$$

Since, $u(P)$ came from the highest bidders by Lemma 1, $m(\{k\}) = a_{kj} = \max_{i\in N} a_{ki}$. □

Consider the following solution: every item is assigned to its highest bidder and the monetary compensations are calculated according to

$$x_i = \sum_{k\in M}\left[c_k \frac{a_{ki}}{\sum_{j\in N} a_{kj}}\right] - u_i,$$

where $c_k = \max_{j\in N} a_{kj}$.

In case of a tie in the highest bid, any mechanism could be used to break it.

One of the main contributions of this work is established in the next theorem: the previous solution is the only one that is Pareto optimal and in proportion with A.

Theorem 1. *The family of solutions* $\{(u(P), x(P))|P \subseteq M\}$ *for* (M, N, A) *is in proportion with* A *and each* $(u(P), x(P))$, $P \subseteq M$ *is Pareto optimal if and only if for every* $P \subseteq M$, $u(P)$ *came from the highest bidders and*

$$x_i(P) = \sum_{k\in P}\left[c_k \frac{a_{ki}}{\sum_{j\in N} a_{kj}}\right] - u_i(P). \tag{2.3}$$

Proof. Suppose a family of solutions $\{(u(P), x(P))|P \subseteq M\}$ for (M, N, A) in proportion with A and where each $(u(P), x(P))$, with $P \subseteq M$ is Pareto optimal. Notice first that for $P = \{k\}$, using (2.2) and the fact that $c_k = \max_{j\in N} a_{kj}$,

$$\varphi_i(\{k\}) = m(\{k\})\frac{a_{ki}}{\sum_j a_{kj}} = c_k \frac{a_{ki}}{\sum_j a_{kj}}. \tag{2.4}$$

Now, for arbitrary $P \subseteq M$ and $i \in N$, by Lemma 2(b) and (c),

$$\varphi_i(P) = \varphi_i(P\backslash\{k\}) + [m(P) - m(P\backslash\{k\})]\frac{a_{ki}}{\sum_{j\in N} a_{kj}} = \varphi_i(P\backslash\{k\}) + c_k \frac{a_{ki}}{\sum_{j\in N} a_{kj}}$$

so, if we repeat the previous argument for the items in $P\backslash\{k\}$ then we get (2.3).

Now, consider the family of solutions $\{(u(P), x(P))|P \subseteq M\}$ for (M, N, A) where $u(P)$ came from the highest bidders and $x(P)$ is given by (2.3). First notice that,

$$(\varphi_i(P) - \varphi_i(P\backslash\{k\}))a_{kj} = c_k \frac{a_{ki}a_{kj}}{\sum_{j\in N} a_{kj}} = (\varphi_j(P) - \varphi_j(P\backslash\{k\}))a_{ki}$$

so, the family $\{(u(P), x(P))|P \subseteq M\}$ of solutions for (M, N, A) is in proportion with A. By Lemma 1, each $(u(P), x(P))$, $P \subseteq M$ is Pareto optimal. This proves the theorem. □

Note that with the solution of Theorem 1 agent i gets $c_k a_{ki}/\sum_{j\in N} a_{kj}$ from item k, so, every agent gets the same percentage $c_k/\sum_{j\in N} a_{kj}$ of the value a_{ki} that he believes the item k has.

Example. Suppose the set $N = \{1, 2, 3\}$ inherits the set of goods $M = \{1, 2\}$ and the agents in N evaluate the goods according to the matrix

$$A = \begin{bmatrix} 25 & 20 & 50 \\ 75 & 180 & 200 \end{bmatrix}$$

Let us calculate the solution established in Theorem 1 for this example. Both goods are assigned to agent 3, since he is the highest bidder in both cases, so agent 3 gets US$ 250 in goods and nothing for the others. Hence, we have that $u^{\mathrm{T}} = (0, 0, 250)$ and the corresponding monetary compensations are given by

$$x = \frac{50}{95}\begin{pmatrix} 25 \\ 20 \\ 50 \end{pmatrix} + \frac{200}{455}\begin{pmatrix} 75 \\ 180 \\ 200 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 250 \end{pmatrix} = \begin{pmatrix} 46.12 \\ 89.64 \\ -135.77 \end{pmatrix}$$

The result is excellent: agent 1 gets US$ 46.12 when he just expected US$ 33.33, agent 2 gets US$ 89.64 when he expected US$ 66.66 and agent 3 gets a benefit of US$ 114.23 (250–135.77) when according to his estimations, he would have received only US$ 83.33. Furthermore, every agent gets the same proportion of what he believes each item has. The next table summarizes the result. The first number in each cell is the value agent receives from the corresponding item and the number in paranthesis is its percentage with respect to the value he gives to the item.

Item	Agent 1 (percentage)	Agent 2 (percentage)	Agent 3 (percentage)
1	13.15 (52.6)	10.52 (52.6)	26.32 (52.6)
2	32.97 (43.96)	79.12 (43.96)	87.91 (43.96)
Total	46.12 (46.12)	89.64 (44.82)	114.23 (45.69)

Now, we consider the case where the agents do not have the same right over the inheritance. Let α_i be the percentage of the inheritance corresponding to agent i, e.g. a husband could leave 60% of the inheritance to his wife and 20% to each of his two sons, so $\alpha_1 = 0.6$, $\alpha_2 = 0.2$ and $\alpha_3 = 0.2$ in this case.

Definition 6. We will say that a family $\{(u(P), x(P))|P \subseteq M\}$ of solutions for (M, N, A) is in a weighted proportion with A if and only if

$$(\varphi_i(P) - \varphi_i(P\backslash\{k\}))\alpha_j a_{kj} = (\varphi_j(P) - \varphi_j(P\backslash\{k\}))\alpha_i a_{ki} \tag{2.5}$$

for every $i, j \in N$ and $k \in P$.

Lemma 3. *If a family $\{(u(P), x(P))|P \subseteq M\}$ of solutions for (M, N, A) is in a weighted proportion with A then*

(a) $\varphi_i(P) = \varphi_i(P\backslash\{k\}) + [m(P) - m(P\backslash\{k\})]\alpha_i a_{ki}/\sum_{j\in N}\alpha_j a_{kj}$, *for every $P \subseteq M$ and $k \in P$.*
(b) $m(P) = \sum_{k\in P} m(\{k\})$.
(c) *Moreover, if the family is Pareto optimal then $m(\{k\}) = \max_{i\in N} a_{ki}$ for every $k \in M$.*

The proof of the lemma is similar to that of Lemma 2.

Theorem 2. *The family of solutions $\{(u(P), x(P))|P \subseteq M\}$ for (M, N, A) is in a weighted proportion with A and each $(u(P), x(P))$, $P \subseteq M$ is Pareto optimal if and only if for every*

$P \subseteq M$, $u(P)$ came from the highest bidders and

$$x_i(P) = \sum_{k\in P}\left[c_k \frac{\alpha_i a_{ki}}{\sum_{j\in N}\alpha_j a_{kj}}\right] - u_i(P) \tag{2.6}$$

The proof of the theorem is similar to that of Theorem 1.

Notice that the solutions of Theorems 1 and 2 could be applied in steps, if we divide the set M, the sum of the solutions of the parts coincide with the solution of the whole set.

Example (Continuation). For the previous example, suppose that agents 1, 2 and 3 have 20, 70, and 10%, respectively of the rights over the inheritance. As before we have that $u^{\mathrm{T}} = (0, 0, 250)$ and the corresponding monetary compensations are given by

$$\begin{aligned} x &= \frac{50}{0.2\times 25 + 0.7\times 20 + 0.1\times 50}\begin{pmatrix} 0.2\times 25 \\ 0.7\times 20 \\ 0.1\times 50 \end{pmatrix} \\ &\quad + \frac{200}{0.2\times 75 + 0.7\times 180 + 0.1\times 200}\begin{pmatrix} 0.2\times 75 \\ 0.7\times 180 \\ 0.1\times 200 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 250 \end{pmatrix} \\ &= \begin{pmatrix} 29.05 \\ 185.689 \\ -214.74 \end{pmatrix} \end{aligned}$$

As before, the amounts agents receive are bigger than they expect. The result of using the solution of Theorem 2 is summarized in the following table

	Agent 1	Agent 2	Agent 3
α	0.2	0.7	0.1
Amount received in goods	0	0	250
Amount received in cash	29.05	185.69	−214.74
Total inheritance's value	100	200	250
Expected assignment	20	140	25
Total assignment	29.05	185.69	35.26
Inheritance's percentage	29.05	92.85	14.10

Now, we modify the definition of being in proportion with A to get a solution where every agent gets the same percentage of the value he believes the whole inheritance has. Moreover, this percentage is the highest that can be guaranteed for all the agents. We denote by $v \in \mathbb{R}^N$ the vector whose j-entry is the value assigned by agent j to the total inheritance, i.e. $v_j = \sum_{i\in M} a_{ij}$.

Definition 7. We will say that the solution (u, x) is in proportion with v if and only if

$$(u_i + x_i)v_j = (u_j + x_j)v_i \tag{2.7}$$

for every $i, j \in N$.

Theorem 3. *The solution* (u, x) *is Pareto optimal and in proportion with* v *if and only if* u *came from the highest bidders and* $x = ((\iota^{\mathrm{T}}u)/(\iota^{\mathrm{T}}v))v - u$ *where* ι *denotes the column vector of all 1's.*

Proof. Suppose a solution (u, x) is Pareto optimal and in proportion with v. By Lemma 1, we just have to prove that $x = ((\iota^{\mathrm{T}}u)/(\iota^{\mathrm{T}}v))v - u$. If we add the equations in (2.7) over $i \in N$, we get

$$(\iota^{\mathrm{T}}u + 0)v_j = (u_j + x_j)\iota^{\mathrm{T}}v$$

and it is follows that, $(\iota^{\mathrm{T}}u/\iota^{\mathrm{T}}v)v_j = u_j + x_j$.

For the converse, suppose $x = (\iota^{\mathrm{T}}u/\iota^{\mathrm{T}}v)v - u$ where u came from the highest bidders. Since,

$$\iota^{\mathrm{T}}x = \frac{\iota^{\mathrm{T}}u}{\iota^{\mathrm{T}}v}\iota^{\mathrm{T}}v - \iota^{\mathrm{T}}u = 0,$$

we see that (u, x) is a solution. Now, from the coordinates i and j of x we get that $(u_i + x_i)/v_i = (u_j + x_j)/v_j$. Therefore (u, x) is in proportion with v. The solution (u, x) is Pareto optimal by the Lemma 1. □

Example (Continuation). Let us calculate the solution established in Theorem 3 for the initial example. As before, both goods are assigned to agent 3, since he is the highest bidder in both cases. Hence, we have that $u^{\mathrm{T}} = (0, 0, 250)$, $v^{\mathrm{T}} = (100, 200, 250)$ and the corresponding monetary compensations are given by

$$x = \frac{\iota^{\mathrm{T}}u}{\iota^{\mathrm{T}}v}v - u = \frac{250}{550}\begin{pmatrix} 100 \\ 200 \\ 250 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 250 \end{pmatrix} = \begin{pmatrix} 45.45 \\ 90.90 \\ -136.36 \end{pmatrix}.$$

so we have,

	Agent 1	Agent 2	Agent 3
Amount received in goods	0	0	250
Amount received in cash	45.45	90.90	−136.36
Total inheritance's value	100	200	250
Inheritance's percentage	45.45	45.45	45.45

Every agent gets the same proportion of what he believes is the value of the inheritance (45.45%) and there is no other solution where the agent with the lowest percent could be higher (see proof of Theorem 6).

Now, we consider the case when the agents do not have the same right over the inheritance. As before, let α_i be the percentage of the inheritance corresponding to agent i.

Definition 8. We will say that the solution (u, x) is in a weighted proportion with v if and only if $(u_i + x_i)\alpha_j v_j = (u_j + x_j)\alpha_i v_i$ for every $i, j \in N$.

We will denote by $\hat{\alpha}$ the diagonal matrix corresponding to the vector $\alpha \in \mathbb{R}^N$.

Theorem 4. *The solution* (u, x) *is Pareto optimal and in a weighted proportion with* v *if and only if* u *came from the highest bidders and* $x = (\iota^{\mathrm{T}} u / \alpha^{\mathrm{T}} v)\hat{\alpha} v - u$.

The proof of Theorem 4 is similar to that of Theorem 3.

3. Some properties and remarks

In this section, we establish some properties of the previous solutions. We start describing four properties that we adapt from Moulin (1992) to the present context: weak individual rationality, resource and population monotonicity and population solidarity. We will say that a procedure has the *weak individual rationality* property when each agent gets with it, at least $1/n$th of what he believes is the value of the whole inheritance. The procedure has a *resource monotonicity* property if the welfare of no agent goes down when the goods to be divided increase and we will say it has the *population monotonicity* property if when we incorporate an additional heir, keeping the same inheritance, none of the agents already present should benefit. Furthermore, we will say the procedure has the *population solidarity* property if when a new agent shows up to share the same goods, the previous agents all benefit or all suffer. A deeper discussion of these properties could be found in Moulin (1992).

Moreover, we will say a procedure is *homogeneous of degree* 1 if a change in the scale of the items's evaluation yield an equivalent change in the total amounts agents receive.

Theorem 5. *Let* (u, x) *be the solution of* (M, N, A) *given by* Theorem 1 *then*

(a) *Weak individual rationality,* $x_i + u_i \geq (1/n)v_i$ *for every* $i \in N$.
(b) *Resource monotonicity, if* $P \subseteq Q \subseteq M$ *then* $\varphi_i(P) \leq \varphi_i(Q)$ *for every* $i \in N$.
(c) *Homogeneous of degree 1, the pair* $(\mu u, \mu x)$ *is a solution of* $(M, N, \mu A)$ *where* μ *is a positive real number.*
(d) *Continuity on* A, *the solution* (u, x) *of* (M, N, A) *is a continuous function of* A.

Proof.

(a) By the additivity of $\varphi_i(M, N)$ in its first coordinate, it is enough consider just one item, say item k. Then

$$x_i + u_i = \frac{c_k}{\sum_{j \in N} a_{kj}} a_{ki} \geq \frac{c_k}{n c_k} a_{ki} = \frac{1}{n} a_{ki}.$$

(b) It follows from the additivity of $\varphi_i(M, N)$ in its first coordinate and by the positivity of A.
(c) It is straightforward.
(d) Continuity follows from the fact that $c_k a_{ki}$ and $\sum_{j \in N} a_{kj}$ are continuous as functions of the a_{ij}'s and by the positivity of the denominator. □

The solution of Theorem 1 does not have the population solidarity property: in the problem with the matrix

$$A = \begin{bmatrix} 10 & 5 & 4 \\ 5 & 10 & 50 \end{bmatrix}$$

agent 2 is better off and agent 1 gets worse when agent 3 leaves.

In the solution of Theorem 1, there is a common percentage

$$\frac{c_k}{\sum_{j\in N} a_{kj}}$$

for each item each agent gets from the value he gives to it and in the solution of Theorem 3 there is a common percentage $\iota^{\mathrm{T}}u/\iota^{\mathrm{T}}v$ each agent gets from the value he gives to the inheritance.

Theorem 6. *The solution* (u, x) *of* Theorem 3 *has the following properties*

(a) *Weak individual rationality,* $x_i + u_i \geq (1/n)v_i$ *for every* $i \in N$.
(b) *Resource monotonicity, if* $P \subseteq Q \subseteq M$ *then* $\varphi_i(P) \leq \varphi_i(Q)$ *for every* $i \in N$.
(c) *Population solidarity, if* $N \subseteq N'$ then $\varphi_i(M, N') \leq \varphi_i(M, N)$ *for every* $i \in N$ *or* $\varphi_i(M, N) \leq \varphi_i(M, N')$ *for every* $i \in N$.
(d) *Homogeneous of degree* 1, *the pair* $(\mu u, \mu x)$ *is a solution of* $(M, N, \mu A)$ *where* μ *is a positive real number*.
(e) *Continuity on* A, *the solution* (u, x) *of* (M, N, A) *is a continuous function of* A.
(f) *There exists a* $\lambda \in \mathbb{R}$ *such that* $u + x = \lambda v$.
(g) $\max_{(u,x)\in\mathcal{S}}\min_{i\in N}(u_i + x_i)/v_i = \lambda$ *where* $\mathcal{S}$ *is the set of solutions*.

Proof.

(a) Clearly $\sum_{l\in N} u_l = \sum_{i\in M} \max\{a_{ik}|\, k \in N\}$, so,

$$\iota^{\mathrm{T}}v = \sum_{j\in N}\sum_{i\in M} a_{ij} \leq \sum_{j\in N}\sum_{i\in M} \max\{a_{ik}|\, k \in N\} = \sum_{j\in N}\sum_{l\in N} u_l = n\,\iota^{\mathrm{T}}u$$

and its follows that

$$\lambda = \frac{\iota^{\mathrm{T}}u}{\iota^{\mathrm{T}}v} \geq \frac{1}{n}.$$

(b) Let us consider two disjoint sets of goods, let u_1, u_2, v_1 and v_2 be the values for the corresponding matrices and suppose λ_1 and λ_2 such that

$$\lambda_1\iota^{\mathrm{T}}v_1 = \iota^{\mathrm{T}}u_1$$

$$\lambda_2\iota^{\mathrm{T}}v_2 = \iota^{\mathrm{T}}u_2$$

without loss of generality, we can suppose $\lambda_2 \geq \lambda_1$ so

$$\frac{\iota^{\mathrm{T}}u_1}{\iota^{\mathrm{T}}u_1 + \iota^{\mathrm{T}}u_2} = \frac{\lambda_1\iota^{\mathrm{T}}v_1}{\lambda_1\iota^{\mathrm{T}}v_1 + \lambda_2\iota^{\mathrm{T}}v_2} \leq \frac{\iota^{\mathrm{T}}v_1}{\iota^{\mathrm{T}}v_1 + \iota^{\mathrm{T}}v_2}$$

So,

$$\frac{\iota^{\mathrm{T}} u_1}{\iota^{\mathrm{T}} v_1} \le \frac{\iota^{\mathrm{T}} u_1 + \iota^{\mathrm{T}} u_2}{\iota^{\mathrm{T}} v_1 + \iota^{\mathrm{T}} v_2}$$

and therefore

$$\frac{\iota^{\mathrm{T}} u_1}{\iota^{\mathrm{T}} v_1} v_1 \le \frac{\iota^{\mathrm{T}} u_1 + \iota^{\mathrm{T}} u_2}{\iota^{\mathrm{T}} v_1 + \iota^{\mathrm{T}} v_2} (v_1 + v_2).$$

(c) There is a common percentage associated with the solution when we consider N and another when we consider N', so all the agents get more with the solution with higher percentage.

(d) It is straightforward.

(e) Continuity follows from the fact that $\iota^{\mathrm{T}} u$ and $\iota^{\mathrm{T}} v$ are continuous as functions of the a_{ij}'s.

(f) If we take $\lambda = \iota^{\mathrm{T}} u / \iota^{\mathrm{T}} v$ then every agent gets $\iota^{\mathrm{T}} u / \iota^{\mathrm{T}} v$ of what he believes is the value of the whole inheritance (v_j).

(g) Suppose a solution $(\tilde{u}, \tilde{x})$ such that

$$\min_{i \in N} \frac{\tilde{u}_i + \tilde{x}_i}{v_i} > \lambda = \frac{\iota^{\mathrm{T}} u}{\iota^{\mathrm{T}} v}$$

then $(\tilde{u}_i + \tilde{x}_i)/v_i > \lambda$ for every $i \in N$. So

$$\sum_{i \in N} \tilde{u}_i = \sum_{i \in N} (\tilde{u}_i + \tilde{x}_i) > \lambda \sum_{i \in N} v_i = \sum_{i \in N} u_i$$

therefore $\iota^{\mathrm{T}} \tilde{u} > \iota^{\mathrm{T}} u$. But this is impossible, because u came from the highest bidders. □

In particular, (g) proves that there is no solution where everybody gets a bigger percentage of the inheritance. Among all solutions, where each player receives the same percentage share of the inheritance in terms of his own valuation, Theorem 3 gives the one with the highest common percentage.

Some of the properties of Theorem 3 still hold for the more general solution of Theorem 4. For example, the percentage agent j gets is no lower than α_j and the solution is homogenous of degree 1.

Formally, we can say that the procedure has the population monotonicity property if $N \subseteq N'$ then $\varphi_i(M, N') \le \varphi_i(M, N)$ for every $i \in N$, but neither the solution of Theorem 1 nor the solution of Theorem 3 has it: Consider $M = \{1\}$, $N = \{1, 2\}$, $N' = \{1, 2, 3\}$ and $A = [2, 3, 20]$ then we get $\varphi(M, N) = (1\frac{1}{5}, 1\frac{4}{5})$ and $\varphi(M, N') = (1\frac{1}{3}, 2, 13\frac{1}{3})$ using the solution of Theorem 1 and $\varphi(M, N) = (1\frac{1}{5}, 1\frac{4}{5})$ and $\varphi(M, N') = (1\frac{3}{5}, 2\frac{2}{5}, 16)$ if we use the solution of Theorem 3. The reason is that agent 3 is the only one that is willing to pay a big amount for the item.

Now, suppose the rules had been explained to the agents and consider the moment they have to declare their item's value. They can lie if they want to, but if one of them says that an item has a lower value than he believes it has, then it might happen that another agent gets the item for a lower price than the one he is willing to pay. And if he says the item has a bigger value then he take a risk of getting the item at this price.

Both procedures could be applied even if some of the agents feel they do not have enough money to pay an overassignment of goods. If one agent does not have enough money, he can invite another person, such as an antique dealer, to present a bid for each item. This agent could present the highest value between what he can afford and the bid of the other person. Following this strategy, agents could hide his preferences from the others and discourage them to manipulate the outcome. Moreover, the agents could invite jointly more people to participate in the process with a corresponding zero percent over the rights of the inheritance.

Implicit in the model, we are assuming that agents value a collection of goods by adding their individual worts and also that they have a quasi-linear utilities, i.e. linear utilities in money.

Besides, both procedures could be used to solve the chore division problem (this problem was posed by Gardner (1978, p. 124)): suppose the agents need to divide a set of chores. In this case, the agents would like to receive the smallest portion of the chores as possible. So, it is enough to request $a_{ij} < 0$ and to interpret $-a_{ij}$ as how much agent j is willing to receive for doing the chore i. Also, the problems of divorce could be solved without further modifications.

A solution is envy-free if no agent thinks that someone else has a better share. Clearly, the solutions presented here do not have this property, but it could be considered an advantage. In a problem with just one item and three or more agents, assuming an envy-free solution we need to assign the same monetary compensation to every agent that does not receive the item. In the case $A = (1, 1000, 1001)$, it is debatable that any couple of agents get the same monetary compensation. However, the procedures have a weaker property, neither the highest bidder envies the other agents, nor any other agent envies the highest bidder.

4. Further reading

Su, F.E., 1999. Rental harmony: Sperner's lemma in fair division. American Mathematical Monthly 106, 930–942.

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References

Brams, S.J., Taylor, A.D., 1996. Fair Division: From Cake-Cutting to Dispute Resolution. Cambridge University Press, Cambridge.

Gardner, M., 1978. Aha! Insight. Freemann, New York.

Moulin, H., 1992. An application of the shapley value to fair division with money. Econometrica 60 (6), 1331–1339.

Raith, M., 2000. Fair negotiation procedures. Mathematical Social Sciences 39, 303–322.
Steinhaus, H., 1946. The problem of fair division. Econometrica 16, 101–104.
Su, F.E., 1999. Rental harmony: Sperner's lemma in fair division. American Mathematical Monthly 106, 930–942.
Tadenuma, K., Thomson, W., 1993. Games of fair division. Working paper no. 351, Rochester Center for Economic Research.
Thomson, W., 1988. The manipulability of the Shapley value. International Journal of Game Theory 17 (2), 101–127.