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Game Theory

A Solidarity Value for n-Person Transferable Utility Games[1]

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Abstract: In this paper, we introduce axiomatically a new value for cooperative TU games satisfying the efficiency, additivity, and symmetry axioms of Shapley (1953) and some new postulate connected with the average marginal contributions of the members of coalitions which can form. Our solution is referred to as the solidarity value. The reason is that its interpretation can be based on the assumption that if a coalition, say S, forms, then the players who contribute to S more than the average marginal contribution of a member of S support in some sense their "weaker" partners in S. Sometimes, it happens that the solidarity value belongs to the core of a game while the Shapley value does not.

1 Introduction and Main Result

Let N be a finite set of n players, called the *grand coalition.* Let Γ denote the linear space of all n-person transferable utility games.

A *value* on Γ is thought of as a vector-valued mapping, say $\varphi:\Gamma\to R^n$, which uniquely determines, for each $v\in\Gamma$, a distribution of the wealth available to the players through their participation in the game v.

Value theory started with the fundamental paper of Shapley (1953) and now takes up a central position in game theory and its applications, especially in economics, social and political sciences.

In this paper, we introduce a *new value* on Γ which reflects some social behavior of players in coalitions. To do this, we define, for any non-empty coalition T and any $v\in\Gamma$, the quantity

$$A^v(T)=\frac{1}{|T|}\sum_{k\in T}[v(T)-v(T\backslash k)],$$

where $|T|$ means the cardinality of T.

Clearly, $A^v(T)$ is the *average marginal contribution* of a member of the coalition T. Next, we define the payoff $\psi_i(v)$ to every player $i\in N$ in any game $v\in\Gamma$ by

$$\psi_i(v)=\sum_{T\ni i}\frac{(n-|T|)!\,(|T|-1)!}{n!}A^v(T). \tag{1.1}$$

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(Recall that $n=|N|$.) We call the mapping $\psi=(\psi_1, \ldots, \psi_n)$, given by (1.1), the *solidarity value* on Γ. Its interpretation can be obtained by replacing the marginal contributions $\mu(T, i):=v(T)-v(T\backslash i)$ of player i in the well-known interpretation of the Shapley value by $A^v(T)$, $\emptyset \neq T \subset N$. To be more specified, if a player i becomes a member of some coalition T, then he/she obtains (as a payoff) the average marginal contribution $A^v(T)$ of a member of T. Thus, if $\mu(T, i)>A^v(T)$, player i offers some part of his/her marginal contribution $\mu(T, i)$ to the coalition T to support some "weaker" members of T. If $\mu(T, i)<A^v(T)$, then player i benefits from the fact that he/she has been accepted to become a new member of the coalition T. Of course, $\psi_i(v)$ is then the expected payoff to player i in the game v and (under the above interpretation) our name for the value ψ is justified. The aim of this paper is to determine the solidarity value axiomatically, but before we describe our axioms, let us look at the following examples.

Example 1.1 (Three Brothers): Players 1, 2 and 3 are brothers and they live together. Player 1 and 2 can make together a profit of one unit, that is, $v\{1, 2\}=1$. Player 3 is a *disabled person* and can contribute nothing to any coalition. Therefore, $v\{1, 2, 3\}=1$. Further, we have $v\{1, 3\}=v\{2, 3\}=0$. Finally, we assume that $v\{i\}=0$ for every player i. This is a classical *unanimity game*. The Shapley value of our game is

$$\Phi(v)=(1/2, 1/2, 0).$$

(Should the disabled brother leave his family?) If players 1 and 2 take the responsibility for their brother (player 3), then the solidarity value

$$\psi(v)=(7/18, 7/18, 4/18)$$

seems to be a "better" solution for the game v than its Shapley value. Of course, one can say that if some kind of solidarity of players 1 and 2 with player 3 is assumed then such a fact should be reflected by the characteristic function v itself. However, the question is then how to define the marginal contributions of player 3 to the grand coalition. The answer is not obvious. We do not want to say that the solidarity value is the "only right" solution concept even for this example. We would rather like to point out that it could be used to take into account some (usually subjective and very difficult to measure) social or psychological aspects in a cooperative game. The characteristic function v might be then used to represent the underlining "pure economic" situations in the game only.

We note that $\psi(v)$ does not belong to the core of v in the above example and the Shapley value $\Phi(v)$ does. We now give another game v for which the solidarity value does belong to the core of v while the Shapley value does not.

Example 1.2: Consider a three person game v where $v\{1\}=v\{2\}=0$, $v\{3\}=1$, $v\{1, 2\}=3.5$, $v\{1, 3\}=0$, $v\{2, 3\}=0$. Finally, $v\{1, 2, 3\}=5$. The Shapley value $\Phi(v)$ of this game is

$$\Phi(v)=(25/12, 25/12, 10/12).$$

Note that $\Phi(v)$ is not individually rational, and thus does not belong to the core of v. The solidarity value $\psi(v)$ of the game v is

$$\psi(v) = (16/9,\ 16/9,\ 13/9),$$

and clearly is in the core of v.

Let φ be a value on Γ. Consider the following axioms.

Axiom A1 *(Efficiency):* For any game $v \in \Gamma$,

$$\sum_{i \in N} \varphi_i(v) = v(N).$$

Axiom A2 *(Additivity):* For any games v, $w \in \Gamma$,

$$\varphi(v+w) = \varphi(v) + \varphi(w).$$

Axiom A3 *(Symmetry):* Let $v \in \Gamma$. For any automorphism π of the game v,

$$\varphi_i(v) = \varphi_{\pi(i)}(v).$$

We remind that π is an automorphism of the game v if $v(\pi(S)) = v(S)$ for each coalition $S \subset N$.

Axiom A4 *(A-null player):* If $i \in N$ is an A-null player in a game $v \in \Gamma$, that is, $A^v(T) = 0$ for every coalition T containing i, then $\varphi_i(v) = 0$.

Axioms A1–A3 are standard. Axiom A4 is new and replaces the null player axiom introduced by Shapley (1953). If it happens that every coalition T containing player i has the average marginal contribution $A^v(T) = 0$, then according to A4 player i gets nothing from the game v.

We can now state our main result.

Theorem: A value $\varphi : \Gamma \to R^n$ satisfies the efficiency, additivity, symmetry and A-null player axioms if and only if $\varphi = \psi$, i.e., φ is the solidarity value.

2 Proof

Using unanimity games, which are helpful in determining the Shapley value, would make the construction of our value very complicated. Therefore, we introduce a *new* basis for Γ, denoted by $\{w_T\}$. For each non-empty coalition T, we define the game w_T to be

$$w_T(S) = \begin{cases} \binom{|S|}{|T|}^{-1} & \text{if } S \supset T, \\ 0 & \text{otherwise}. \end{cases} \tag{2.1}$$

Lemma 2.1: For any non-empty coalition T, the game w_T has the following properties:

(i) $w_T(T) = 1$;
(ii) If $S = T \cup E$ with $\emptyset \neq E \subset N \backslash T$, then

$$w_T(S) = \frac{1}{|S|} \sum_{i \in S} w_T(S \backslash i), \tag{2.2}$$

and every player $i \in N \backslash T$ is A-null in the game w_T.

Proof: (i) is obvious. To show (ii) take any $S = T \cup E$, where $T \cap E = \emptyset$ and $E \neq \emptyset$. Then using (2.1) we get

$$\begin{aligned} w_T(S) - \frac{1}{|S|} \sum_{i \in S} [w_T(S \backslash i)] &= \binom{|S|}{|T|}^{-1} - \frac{1}{|S|} \sum_{i \in E} [w_T(S \backslash i)] \\ &= \frac{|T|!\,|E|!}{|T \cup E|!} - \frac{1}{|T \cup E|} |E| \frac{|T|!(|E|-1)!}{(|T \cup E|-1)!} = 0, \end{aligned}$$

which gives (2.2). Let $i \in N \backslash T$. Then for $v = w_T$ and for each coalition S containing player i, we have $A^v(S) = 0$. This is obvious if T is not a subset of S. If $T \subset S$, then $A^v(S) = 0$ follows from (2.2). Thus, i is an A-null player in the game w_T. ∎

Lemma 2.2: The family $\{w_T\colon T \subset N,\ T \neq \emptyset\}$ of games defined by (2.1) is a basis for the linear space Γ.

Proof: Put $K = 2^n - 1$. It is known that Γ is a K-dimensional linear space. Let S_1, $S_2, \ldots, S_K$ be a fixed sequence containing all non-empty subsets of N such that

$$n = |S_1| \geq |S_2| \geq \ldots \geq |S_K| = 1.$$

Further, let $A = [a_{ij}]$ be the $K \times K$ matrix defined by

$$a_{ij} = w_{S_i}(S_j), \quad i, j = 1, 2, \ldots, K.$$

It follows from (2.1) that A is a triangle matrix with all diagonal entries equal to 1. Hence $\det A \neq 0$, and this immediately implies that the games $\{w_{S_i}\colon i = 1, 2, \ldots, K\}$ constitute a set of K independent vectors in the linear space Γ, and thus, a basis for Γ. ∎

Lemma 2.3: If $\varphi=(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is a value on Γ satisfying the efficiency, additivity, symmetry and A-null player axioms, then for every player $i\in N$, non-empty coalition T, and any constant c,

$$\varphi_i(cw_T) = \begin{cases} c\begin{pmatrix} |n| \\ |T| \end{pmatrix}^{-1} / |T| & \text{if } i\in T, \\ 0 & \text{if } i\in N\setminus T. \end{cases} \tag{2.3}$$

Proof: Fix any non-empty coalition T. If $c=0$, then the lemma follows immediately from the symmetry and efficiency axioms. Assume that $c\neq 0$. From Lemma 2.1, we conclude that every player $i\in N\setminus T$ is A-null in the game cw_T. Thus, $\varphi_i(cw_T)=0$ for all $i\in N\setminus T$, and the remaining part of (2.3) follows now from the efficiency and symmetry axioms. ■

From A2 and Lemmas 2.2 and 2.3, we conclude the following simple fact.

Lemma 2.4: Any value satisfying axioms A1–A4 is a linear mapping from Γ into R^n.

Proof of Theorem: The existence of a value which satisfies our axioms A1–A4 is quite obvious. Clearly, ψ given by (1.1) satisfies the symmetry and A-null player axioms. Moreover, ψ is a linear mapping. Hence A2 is satisfied. To show the efficiency first note (using Lemma 2.1) that ψ is efficient for any base game w_T. If $v\in\Gamma$, then there exist constants λ_T, $\emptyset\neq T\subset N$, such that

$$v = \sum_{\emptyset\neq T\subset N} \lambda_T w_T.$$

Using now linearity of ψ, we get

$$\sum_{i\in N} \psi_i(v) = \sum_{\emptyset\neq T\subset N} \lambda_T \sum_{i\in N} \psi_i(w_T) = \sum_{\emptyset\neq T\subset N} \lambda_T w_T(N) = v(N),$$

which proves that ψ is an efficient value.

To prove the uniqueness, consider a value φ on Γ which satisfies A1–A4. By Lemma 2.4, φ is a linear mapping. Applying Lemma 2.3 to both φ and ψ, we infer that $\varphi(w_T)=\psi(w_T)$, for each base game w_T. Thus, $\varphi(v)=\psi(v)$ for every game $v\in\Gamma$. ■

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References

Shapley LS (1953) A value for n-person games. In: Kuhn HW, Tucker AW (eds) Contributions to the Theory of Games II, Annals of Mathematics Studies, Princeton University Press, Princeton 307–317

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